Matrix of α and β

The square of all the four matrix so their eigen values are +1 and −1, Let us arbitratily choose β as the matrix that is to be diagonal and we rearrange its rows and columns so that all the +1 eigen values are group together in a matrix of rank n and all −1 eigen values are grouped together in a matrix of ramk m.

The matrix β can be expressed as.

$$\beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\beta = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

All the four Matrix α_x, α_y, α_z and β having their square is unity and they anticommute with one another in pairs. We already have three well known 2×2 matrices σx, σy, σz called Pauli spin matrices which satisfied above properties

$$\sigma x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Since a 2x2 matrix has four elements, there are four and only four independent 2x2 matrices three of these are $\sigma x, \sigma y, \sigma z$. The only other matrix linearly independent.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Which is unit matrix therefore commute rather than anticommutes with every σ, hence can't have fourth 2x2 matrix which satisfies both properties of Dirac Matrices. Now we show that the Dirac Matrix must be even dimensional.

Let us choose a representation in which β is diagonal NxN matrix i.e.

$$\beta = \begin{bmatrix} b_1 & & 0 \\ & b_i & \\ 0 & & b_N \end{bmatrix} \quad \text{—— (1)}$$

As $\beta^2 = 1$, $b_i^2 = 1$ and $b_i = \pm 1$

Also $\beta^2 = \alpha_K^2 = 1$ $(K = x, y, z)$, $\det \alpha_K$ ar $\det \beta \neq 0$ this implies that matrices $(K = x, y, z)$ and β has an inverse.

Since β anticommutes with each other α (3)

$$\alpha_K \beta + \beta \alpha_K = 0$$

This relationship may be express as

$$\beta \alpha_K = - \alpha_K \beta$$

$$\alpha_K^{-1} \beta \alpha_K = - \alpha_K^{-1} \alpha_K \beta$$

As $\alpha_K^{-1} \alpha_K = 1$

$$\alpha_K^{-1} \beta \alpha_K = - \beta$$

taking trace of both side

$$\text{Trace}(\alpha_K^{-1} \beta \alpha_K) = - \text{trace}(\beta)$$

$$\text{Trace}(\alpha_K \alpha_K^{-1} \beta) = - \text{trace}(\beta)$$

$$[\text{Trace}(ABC) = \text{Trace}(CBA)]$$

$$\text{Trace}(\beta) = - \text{Trace}(\beta)$$

$\text{Trace}\ \beta = 0$, Similarly ~~as~~ $\text{Trace}(\alpha_K) = 0$

$$\text{Trace}(\beta) = \text{Trace}(\alpha_K) = 0$$

In matrix 1, Let r of b_i's $+1$ rest are -1

$$b_1 = b_2 = \ldots = b_r = 1$$

$$b_{r+1} = b_{r+2} = b_N = 1$$

$r + s = N$, but ~~trace~~ $\text{Trace}(\beta) = 0$, requires

$\sum_{i=1}^{N} b_i = r - s = 0$, i.e $r = s$

$\boxed{N = 2r}$. The next simplest choice 4×4

we can't use 3×3

Dirac matrix α and β must be even dimensional,